E-968

DESIGN OF A DISH ANTENNA FOR PRESCRIBED DEFORMATIONS

by

John M. Dahlen

November, 1960

INSTRUMENTATION LABORATORY

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Cambridge 39, Mass.

62-1-5

558

E-968

DESIGN OF A DISH ANTENNA FOR PRESCRIBED DEFORMATIONS

by

John M. Dahlen

November, 1960

INSTRUMENTATION LABORATORY
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
CAMBRIDGE 39, MASSACHUSETTS

Approved: Philip N. Bowditch
Assistant Director

ACKNOWLEDGEMENT

This report was prepared under the auspices of DSR Project 52-156, sponsored by the Ballistic Missile Division of the Air Research and Development Command through USAF contract AF 04(647)-303.

ABSTRACT

A method is developed for sizing the structural truss members of a dish antenna [redacted] to assure retention of paraboloidal form and constant focal length during external loading. This method can be modified to suit other structural forms and deformation requirements. Since this method is most easily applied to statically determinate structures, a short discussion on the synthesis of rigid, firmly anchored, statically determinate trusses is given.

TABLE OF CONTENTS

LIST OF SYMBOLS

b	number of bars
r	support force components
s	number of support points
j	number of socket joints
X	deflection in the X-direction
$\overline{F}$	external force per unit volume
$\overline{C}$	appropriate influence function
Y	deflection in Y-direction
Z	deflection in Z-direction
$\left\{\frac{1}{S}\right\}$	reciprocal of the unknown bar cross-sectional areas
R	radial deflection
Z_i, R_i	axial and radial deflection respectively at point i
F_{z_j}, F_{r_j}	axial and radial components, respectively of the external force applied at point j
P_k	tensile or compressive load in bars k
L_k	length of bars k
S_k	cross-sectional area of bars k
n_k	number of bars of type k
n_i, n_j	number of points of type i or j
U_k	elastic strain energy stored in bar k
l	distance from vertex to focal point of dish
E	Young's Modulus
δ_k	axial deflection of each bar
T_z, T_r	load coefficients, axial and radial respectively
H_k	defined by eq. (8) p. 23
A	beam cross-sectional area

INTRODUCTION

Recent investigations of dish antenna design innovations at the M. I. T. Instrumentation Laboratory have included studies into the benefits of controlling structural deformations under gravity and wind loads. For example, the idea of heating the members of a truss structure to hold their lengths constant under slowly varying loads has shown considerable promise. * Such a control system would be ineffective against rapidly varying loads, such as those due to wind gusts, because of the thermal lag of the structure.

As a possible approach to this latter problem, it was decided to study the feasibility of sizing the structural members relative to each other so that, under given loading conditions, the paraboloidal shape would be retained. It is clearly impossible to design a structure which can maintain its shape under a wide variety of load configurations; however, the idea of controlling deformations by properly sizing the structural members relative to each other has a practical application in two rather general design situations:

a. The situation in which the external loads can be restricted to act primarily in one configuration with respect to the structure.

b. The situation in which the external loads are mainly predictable with respect to configuration and frequency of occurrence. In this case, using statistical methods, the structure can be sized to minimize the standard deviation of some selected deformation-induced error.

* Gras, Ranulf W. , An "Infinite Stiffness" Structural Member Dimensional Stabilization of a Variable - Load Structural-Member, Report R-271, (Instrumentation Laboratory, Massachusetts Institute of Technology, Cambridge, Massachusetts, March 1960).

This report develops a method for treating the ball-jointed truss structure, and follows through to a solution for only the symmetrical loading configuration which results from a wind blowing parallel to the axis of the paraboloid. However, the discussion points out the means for performing the modifications required to extend the method to other structural forms.

DESCRIPTION OF THE PROBLEM

The antenna configuration to be treated has been selected for the attention of structural and thermal control system design studies. While not optimum, this configuration is sufficiently realistic to serve as a valid example for a test solution as well as a model with which to illustrate the analytical procedures.

Figure 1 shows the essential features of a ball-jointed truss whose function is to support the reflecting surface material. Note that the geometrical configuration consists of nine equal truss sectors connected to a center post. The dish is 60 feet in diameter and has a focal point 24 feet from the vertex of the paraboloid. Table I provides detailed dimensional data. The bar types are numbered 1 through 12, there being eighteen each of types 1, 2, 3, 4, and 6, and nine each of the other seven types. Note also that joints (1) through (6) are contained in the paraboloidal surface and are repeated nine times each. The bars connecting these points are shown by solid lines. The bars running from the surface of the dish to points (7) located behind the dish are shown by dashed lines. In Fig. 1 the arrangement of bars in a plane containing the dish axis and a single joint of each type is also shown.

A critical design feature, the method by which one should attach the antenna truss to the base, cannot be determined without knowledge of the antenna application. This feature, in addition to the connections with the center post and the type of truss joints, will define whether or not the structure is rigid and statically

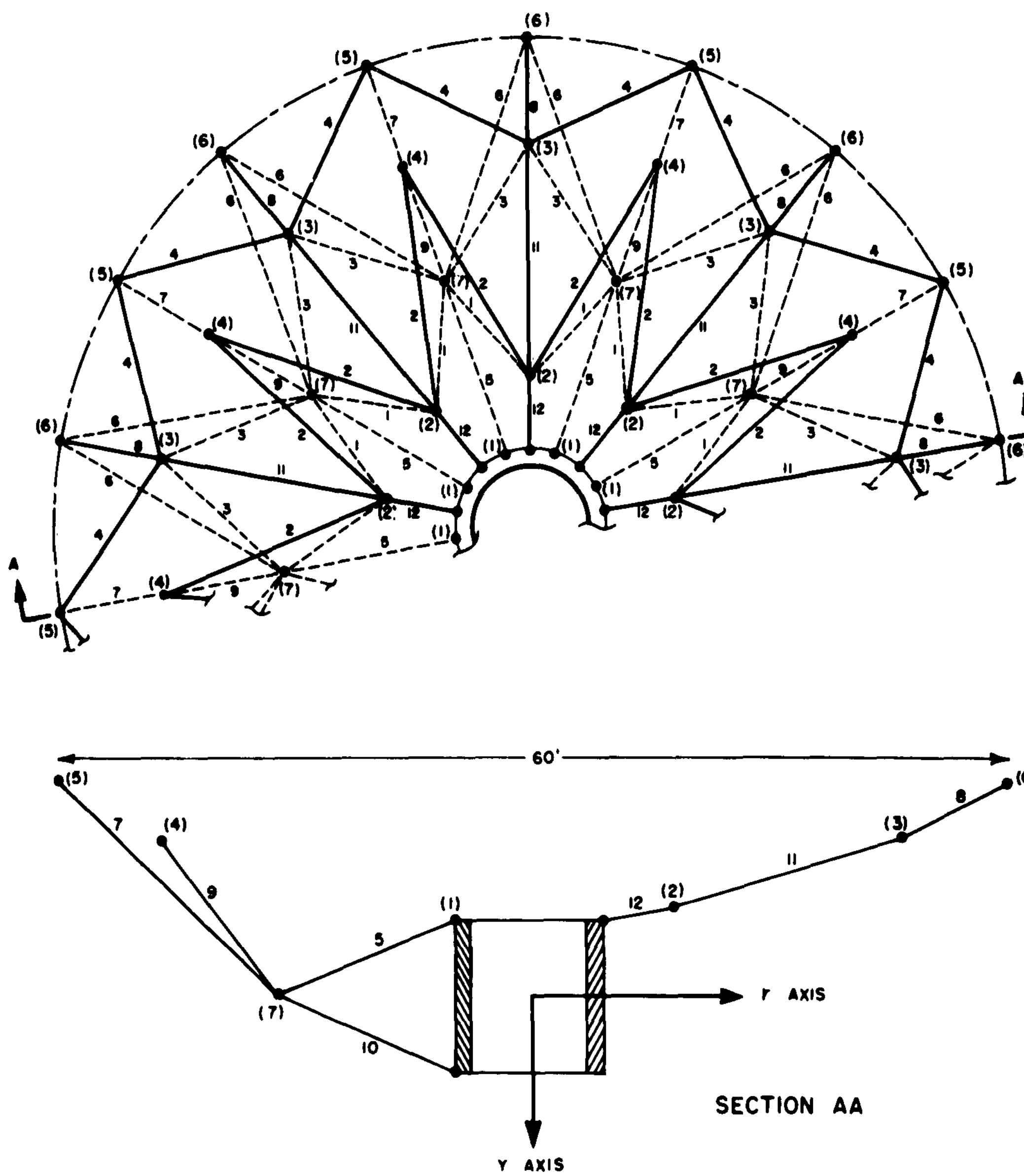


Fig. 1 Configuration of Sample Antenna

Table I

Dimensional Data for Sample Antenna

Table Ia   Bar Lengths and Angles from Dish Axis

Bar Type	Length (Ft)	Angle from Dish Axis (deg)
1	10.242	55.90
2	14.902	72.42
3	13.699	41.60
4	12.141	71.37
5	13.015	67.40
6	20.722	47.00
7	19.178	42.62
8	8.016	61.12
9	11.853	30.22
10	13.015	67.40
11	13.904	71.13
12	4.882	81.22

Table Ib   Angles Projected in Plane Normal to Dish Axis

(Note: For example, the angle 4(3)8 is formed at joint (3) between the projections of bars 4 and 8 in the plane normal to the dish axis.)

Angle	Degrees
8 (6) 6	22.6
11 (3) 3	39.8
8 (3) 4	63.1
11 (2) 2	33.7
11 (2) 1	43.4
5 (7) 1	23.4
9 (7) 6	42.6
9 (7) 3	59.8
9 (4) 2	13.7
7 (5) 4	43.1

determinate. A statically determinate structure is one whose load distribution and support reactions can be calculated from external forces using only force-balance equations. For such a structure, the number of unknown loads and support reactions equals the number of independent force-balance equations. A rigid truss of this type can be assembled on a pre-fabricated foundation with pre-cut bars of imperfect length and straightness. Its joints and support points must be of a type which will permit connection of these imperfect members to each other and to the supports without inducing stresses. The effect of imperfect sizing will modify the geometry only slightly. A firmly anchored, rigid, three-dimensional truss having j-ball and socket joints, s-support points, r-generally non-zero support reaction force components, and b-bars must satisfy the relation,

$$b + r = 3(j + s) \tag{1}$$

in order to be statically determinate. This equation, which is a necessary but not sufficient condition, results from the fact that three force-balance equations can be written at each joint and support point in order to solve for b-axial bar loads and r-support force components. The above relation also assumes that the support points cannot exert bending moments on the bars which are connected to them. They may be of the ball-on-a-plane, ball-in-a-groove or ball-in-a-socket type for which r equals one, two or three, respectively.

A determinate structure has the advantages of rapid assembly from prefabricated parts and stress-deformation analyses which are at once valid and straightforward. However, an indeterminate structure (for example, one using riveted or welded joints and many support points), while difficult to analyze and construct, may offer somewhat higher stiffness-to-weight ratios. This advantage does not appear significant when the thermal control system and prescribed deformation concept can be effectively applied.

In the analysis made in this report, the unresolved antenna design problems cited above are conveniently side-stepped by assuming a symmetrical external load configuration and a symmetrical structure.

The symmetrical load will be assumed to be that due to a wind along the dish axis which will exert axial and radial force components on the dish surface that are functions only of the distance from the axis. No torque will be exerted on the center post. Therefore the question of how to restrain rotations of the center post with respect to the truss may be left open. Similarly, it will be assumed that the antenna is supported on all points (7) which can roll on one plane or in co-planar radial grooves. The symmetry of the truss and external load will then permit the assumption of equal reaction forces at all points (7).

To illustrate how one might configure a rigid, determinate antenna structure which can be firmly anchored, consider the center post to be a rigid support column. To prevent rotation of the truss around this column, replace the nine bars 12 by nine pairs of bars, each pair running from a joint (2) to the two nearest joints (1). Let all joints and support points be of the ball and socket type. Then $b = 162$, $j = 54$, $s = 18$ and $r = 3 \times 18 = 54$. Note that Eq (1) (a necessary but not sufficient condition) is satisfied.

One might also configure a suitable structure resting on ball and socket supports at all points (7) ($s = 9$, $r = 27$). For example, the center post and all bars 5 and 10 could be eliminated. All bars 12 would then come together at a single center joint constituting the vertex of the paraboloid. This vertex would then need to be restrained from axial motion by joining it to three of the support points with three equally spaced bars. A symmetrical structure having 138 bars and 46 ball joints would result. Again, the necessary relationship, Eq (1), would be satisfied.

The determinate truss has been stressed in this section because the analytical work which follows applies to this kind of structure, for which the loads can be calculated in advance of any decisions with respect to the sizing of the individual bar cross-sectional areas. It will be shown, however, that the method given is readily extended to indeterminate structures at the expense of greater complexity.

ANALYTICAL APPROACH

The problem of design for prescribed deformations has been approached as follows:

1) Given a linear, elastic structure of specific geometry and subjected to a load distribution of prescribed form, it is observed that:

   a. The form of the deflection distribution is a function only of the form of the stiffness distribution. In the determinate truss structure this form is manifested in the relative cross-sectional areas of the bars.

   b. The total deflection or amplitude of the deflection distribution is determined by the amplitudes of the load and stiffness distributions and the length of a representative dimension. In other words, for the antenna truss, the diameter of the dish and the actual cross-sectional area of any bar determine the scale factor relating the amplitude of the deflection distribution to the amplitude of the load distribution.

For the antenna problem, any convenient diameter, load distribution magnitude and reference bar cross-sectional area may be used. The solution will be in the form of a set of bar cross-sectional areas which assure the desired deformational form under the prescribed loading configuration.

2) Next it is noted that for a linear, elastic structure, the deflection at any point can be written as a linear combination of the external loads at all points on the structure. Most generally this relationship can be expressed by a Fredholm integral equation of the first kind which takes the form:

$$X(x, y, z) = \iiint \bar{C}(x, y, z; \xi, \eta, \zeta) \cdot \bar{F}(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta \qquad (2)$$

For example if, X were the deflection in the X direction at (x, y, z), $\bar{F}$ would be the external force per unit volume and $\bar{C}$ the appropriate influence function.

For the antenna truss, which is loaded at discrete points with concentrated loads, it is more appropriate to employ the matrix form of the integral equation:

$$\{X\} = [C_{xx}]\{F_x\} + [C_{xy}]\{F_y\} + [C_{xz}]\{F_z\} \quad (3)$$

Then $\{X\}$ would be a column matrix whose i elements represent the deflection in the direction of the x axis at i selected points. The force component column vectors, $\{F_x\}$, $\{F_y\}$ and $\{F_z\}$, would each have j elements expressing the x, y, and z components, respectively, of the external forces at the j loading points. The influence coefficient matrices would then be rectangular and of order i by j. From the discussion given above, it is expected that the matrix equations will express the desired deflection / bar cross-sectional area relationships.

3) Finally, it remains to determine the stiffness distribution which will result in the prescribed functional form of the deflections. That is, the satisfaction of one or more relations of the following form is required:

$$f(X, Y, Z) = 0 \qquad (4)$$

The functions, X, Y, and Z are taken to represent the deflections in the x, y, and z directions, respectively.

In the general case this final step requires the solution of integral equations for the necessary influence functions, which in turn lead to the stiffness distribution.

Fortunately, in the truss example, the solution is far more easily arrived at. The matrix equations, such as Eq (3) which relates deflection to force, will, after computation of influence coefficient matrices, reduce to the following form which expresses each deflection as a linear combination of the reciprocals of the unknown bar cross-sectional areas:

$$\{X\} = [K_x]\left\{\frac{1}{S}\right\}$$

$$\{Y\} = [K_y]\left\{\frac{1}{S}\right\}$$

$$\{Z\} = [K_z]\left\{\frac{1}{S}\right\}$$

Substitution of these equations into the relationships such as Eq (4) required to exist among the deflections themselves will result finally in a set of simultaneous equations which may be solved for the necessary cross-sectional areas.

DEVELOPMENT OF EQUATIONS

In this section, the equations applicable to the symmetrically loaded antenna truss will be developed. It is convenient to use fixed polar coordinates centered at the intersection of the dish axis with the plane containing the support points. The z-axis is taken coincident with the dish axis and pointing toward the rear. The r-axis is then located in the plane of the support points. Axial and radial deflections are then denoted by Z and R, respectively. Table II defines the terminology used. The first step will be to develop the force/deflection equations, a task which mainly consists of the determination of the influence coefficient matrices. The final equations are then obtained by substituting these results into the prescribed deformation relationships.

1. Force/Deflection Equations

The discussion of the analytical approach, already given, indicates for the symmetrical condition force/ deflection equations of the form:

$$\begin{aligned} \{Z\} &= [C_{zz}]\{F_z\} + [C_{zr}]\{F_r\} \\ \{R\} &= [C_{rz}]\{F_z\} + [C_{rr}]\{F_r\} \end{aligned} \qquad (5)$$

Table II

Terminology Undefined in the Text for the Antenna Truss Equations

Z_i, R_i	axial and radial deflections respectively at point i
F_{z_j}, F_{r_j}	axial and radial components, respectively, of the external force applied at point j.
P_k	tensile or compressive load in bars k
L_k	length of bars k
S_k	cross-sectional area of bars k
n_k	number of bars of type k
n_i, n_j	number of points of type i or j, respectively
U_k	elastic strain energy stored in bar k
l	distance from vertex to focal point of dish

In examining a typical element of an influence coefficient matrix, for example, $C_{zr_{ij}}$, this term is recognized to be the influence coefficient which gives the z deflection at point i, Z_i, due to a unit radial external force at point j.

For the symmetrical problem, the algebra may be simplified by considering groups of similar points. If, for example, each joint in the truss were to be considered individually, i and j would take on all positive integral values up to 72. Furthermore, the calculation of individual influence coefficients would be more laborious because a single load at one point would cause unsymmetrical deformation. In the present case, however, it is known that all points of the same type will be subjected to equal loads and equal deflections. Hence, $C_{zr_{ij}}$ will be considered to be the <u>z deflection of each of the nine joints of type i due to unit radial loads at each of the nine joints of type j.</u>

A similar form of definition, of course, will apply to the elements of the other three influence coefficient matrices.

2. <u>Calculation of the Influence Coefficients</u>

The most direct approach to the calculation of $C_{zr_{ij}}$ might be considered the following:

a. Calculate all bar loads P_k due to the application of unit external radial forces at all points j.

b. Knowing all P_k, calculate the axial deflection of each bar (call this δ_k) from the well-known formula:

$\delta_k = P_k L_k / S_k E$ (E = Young's Modulus)

c. Using geometrical considerations find the deflection of all joints in the truss.

This procedure will yield $C_{zr_{ij}}$ and $C_{rr_{ij}}$ for all i. A close inspection of the antenna configuration will, however, convey a distinct sense of the difficulties associated with step c.

A more convenient approach utilizes the energy conservation principle in equating the work done on the structure by external forces with the strain energy stored in the deformed structure. The formal statement of this principle applied to linear, elastic structures is known as Clapeyron's Law. The elastic strain energy due to tension or compression in a typical bar is given by:

$$U_k = \frac{1}{2}\left(\frac{S_k E}{L_k}\right)\delta_k^2 = P_k^{\,2} L_k / 2S_k E$$

Note that $S_k E / L_k$ is the bar stiffness constant expressing the ratio of force to deflection. The total strain energy stored in the antenna truss under symmetrical loading is given by:

$$U = \frac{1}{2E}\sum_k n_k L_k P^2{}_k / S_k$$

The bar loads, P_k, can be calculated in terms of load coefficients $T_{z_{kj}}$ or $T_{r_{kj}}$ which are defined as the loads in all bars of type k due to unit axial or radial external forces, respectively, applied at all nine joints of type j. These load coefficients are positive for tensile bar loads and negative for compressive bar loads. They are calculated from a structural load analysis which, for determinate structures, can be carried out beforehand without regard to deflection considerations. For example, $T_{z_{kj}}$ can be found for all k by calculating all the antenna bar loads due to unit external axial loads applied simultaneously at all nine joints j. The three independent static force balance equations are then solved at each joint. This tedious procedure was carried out on the IBM 650 computer in a few minutes for the sample problem. Table III contains the computed load coefficients. Calculation of the total bar loads from the load coefficients is conveniently expressed by the matrix equation:

$$\{P\} = [T_z]\{F_z\} + [T_r]\{F_r\} \tag{6}$$

Now one may apply Clapeyron's Law to find the influence coefficients. Assume the antenna is first loaded with unit axial loads at all points j. The work done by these loads is given by:

Table III

Load Coefficients Computed for the Sample Antenna Truss

(computer data rounded off to two decimal places)

$[T_{z_{kj}}]$

k/j	1	2	3	4	5	6	7
1	0	-.77	.13	.11	.09	.25	0
2	0	0	0	.39	0	0	0
3	0	0	-.87	0	.35	.40	0
4	0	0	0	0	1.16	0	0
5	-1.30	.44	-.21	-.33	-.67	-.59	0
6	0	0	0	0	0	-1.62	0
7	0	0	0	0	-2.36	0	0
8	0	0	0	0	0	2.49	0
9	0	0	0	-1.43	0	0	0
10	1.30	.81	-.63	-.63	-.97	-1.42	0
11	0	0	.94	0	.67	1.87	0
12	0	-.93	1.06	.75	.75	2.10	0

$[T_{r_{kj}}]$

k/j	1	2	3	4	5	6	7
1	0	-.12	.19	.18	.10	.26	0
2	0	0	0	.67	0	0	0
3	0	0	-.30	0	.38	.40	0
4	0	0	0	0	1.26	0	0
5	0	.27	.05	-.02	-.18	-.16	.54
6	0	0	0	0	0	-.89	0
7	0	0	0	0	-1.09	0	0
8	0	0	0	0	0	2.52	0
9	0	0	0	-.47	0	0	0
10	0	-.08	-.57	-.53	-.51	-1.01	.54
11	0	0	1.38	0	.73	1.89	0
12	0	.87	1.55	1.29	.82	2.13	0

$$\frac{1}{2} n_j C_{zz_{jj}}$$

The factor of 1/2 exists in this term because, to prevent imparting kinetic energy to the truss, the loads are applied slowly, thus causing a linear load-deflection curve. The work input, which is given by the integral of force with respect to displacement in the direction of the force, is the area under this curve which increases linearly from the origin to the point $(n_j, C_{zz_{jj}})$. Equating work input to stored energy, one obtains:

$$\frac{1}{2} n_j C_{zz_{jj}} = \frac{1}{2E} \sum_k (n_k L_k / S_k) T_{z_{kj}}^2$$

If instead the antenna had been loaded with radial loads at all points j, the same expression would have been obtained with z replaced by r.

If one continues to apply the unit axial loads at points j and now adds unit axial loads at all points i, additional work is done on the structure of the amount:

$$\frac{1}{2} n_i C_{zz_{ii}} + n_j C_{zz_{ji}}$$

The first term is the work done by the new loads. The second term is the work done by the original loads while they move through displacements caused at points j by the new loads at points i. Clapeyron's Law applied to the new situation then requires:

$$\frac{1}{2} n_j C_{zz_{jj}} + \frac{1}{2} n_i C_{zz_{ii}} + n_j C_{zz_{ji}}$$

$$= \frac{1}{2E} \sum_k \left(n_k L_k / S_k \right) (T_{z_{kj}} + T_{z_{ki}})^2$$

Reversing the procedure by applying loads first at i and then at j would, of course, result in the same strain energy. The work done would in this case be expressed by the left-hand side of the above equation with the subscripts i and j reversed. Hence, it would be evident that,

$$n_j C_{zz_{ji}} = n_i C_{zz_{ij}}$$

Note also that the above equations permit calculation of $C_{zz_{ij}}$ and $C_{rr_{ij}}$ explicitly.

A similar argument tracing the work inputs when the truss is loaded, first axially at points i and then radially at points j, leads to the equation:

$$\frac{1}{2} n_i C_{zz_{ii}} + \frac{1}{2} n_j C_{rr_{jj}} + n_i C_{zr_{ij}}$$
$$= \frac{1}{2E} \sum_k (n_k L_k / S_k)(T_{z_{ki}} + T_{r_{kj}})^2$$

Reversing the loading procedure by applying the radial loads at points j first leads to the equality,

$$n_i C_{zr_{ij}} = n_j C_{rz_{ji}}$$

This equation, which has already been proved valid when z replaces r (or vice-versa) is a manifestation of Maxwell's Law of Reciprocal Deflections, a corollary of Clapeyron's Law.

A general expression for the influence coefficients is then given by the following equation:

$$C_{zr_{ij}} = \frac{1}{En_i} \sum_{k=1}^{12} \frac{n_k L_k}{S_k} T_{z_{ki}} T_{r_{kj}} \qquad (7)$$

In this equation one merely substitutes r for z and/or z for r whenever necessary in order to express the desired coefficient.

3. Deflection / Bar Stiffness Equations

Substitution of the explicit solution just found for the influence coefficients into the force/deflection equations (5) leads to the following:

$$Z_i = \frac{1}{En_i} \left(\sum_j F_{z_j} \sum_k \frac{n_k L_k}{S_k} T_{z_{ki}} T_{z_{kj}} \right.$$

$$\left. + \sum_j F_{r_j} \sum_k \frac{n_k L_k}{S_k} T_{z_{ki}} T_{r_{kj}} \right)$$

$$R_i = \frac{1}{En_i} \left(\sum_j F_{z_j} \sum_k \frac{n_k L_k}{S_k} T_{r_{ki}} T_{z_{kj}} \right.$$

$$\left. + \sum_j F_{r_j} \sum_k \frac{n_k L_k}{S_k} T_{r_{ki}} T_{r_{kj}} \right)$$

Reversing the order of summation and recognizing the factor,

$$P_k = \sum_j (T_{z_{kj}} F_{z_j} + T_{r_{kj}} F_{r_j}) \quad ,$$

one finds:

$$Z_i = \frac{1}{En_i} \sum_k \frac{n_k L_k P_k}{S_k} T_{z_{ki}}$$

$$R_i = \frac{1}{En_i} \sum_k \frac{n_k L_k P_k}{S_k} T_{r_{ki}}$$

The chosen problem is only concerned with the loads and deflections at points (1) through (7) inclusive. Hence, n_i is a constant equal to 9. Also note that k takes on all positive integral values from 1 through 12, inclusive. It is convenient to define a new variable,

$$H_k = \frac{n_k L_k}{n_i E S_k} \tag{8}$$

which has the dimensions of a compliance coefficient and which contains the unknown bar cross-sectional area explicitly. Then:

$$\left.\begin{aligned} \{Z\} &= [T_z]^T \{P_k H_k\} \\ \{R\} &= [T_r]^T \{P_k H_k\} \end{aligned}\right\} \tag{9}$$

(Note that the superscript T denotes the transpose of the matrix.)

4. <u>Prescribed Deformation Relationships</u>

The paraboloidal surface of the antenna is given by:

$$z = z\,(0) - r^2/4\ell$$

The requirement that the focal length remain constant after deformation dictates that the deformed surface be given by:

$$z + Z = z(0) + Z(0) - (r + R)^2/4$$

Taking the difference between these two equations and neglecting the higher order R^2 term, one obtains the functional relationship which must be satisfied by the deformed truss in order to maintain constant focal length:

$$Z - Z(0) + rR/2\ell = 0 \qquad (10)$$

In terms of the six joint types on the dish surface, Eq (10) requires:

$$\left.\begin{aligned}
&Z_2 - Z_1 + r_2 R_2/2\ell = 0\\
&Z_3 - Z_1 + r_3 R_3/2\ell = 0\\
&Z_6 - Z_1 + r_6 R_6/2\ell = 0\\
&Z_3 - Z_4 = 0\\
&R_3 - R_4 = 0\\
&Z_5 - Z_6 = 0\\
&R_5 - R_6 = 0
\end{aligned}\right\} \qquad (11)$$

5. Final Equations

Substitution of the deflection/bar stiffness equations, Eq (9), into the prescribed deformation equations, Eq (11), will now yield seven linear algebraic equations. Since there are twelve unknown bar areas, it is clear that one may, in principle at least, arbitrarily set five bar areas and solve for the remaining seven. In the sample problem, it was decided to set the cross-sectional areas of bars 8 through 12 so that they would undergo equal stress. Since these bars were the ones carrying the greatest loads, this decision seemed reasonable. It was then required that:

$$\left|P_k/S_k\right| = \left|P_8/S_8\right| \text{ for } k = 9,\ 10,\ 11,\ 12$$

Recalling the earlier discussion that only the relative bar areas need to be considered, H_8 is arbitrarily set equal to 1. It then follows that:

$$H_k = \frac{n_k}{n_8}\ \frac{L_k}{L_8} \left| \frac{P_8}{P_k} \right| \text{ for } k = 9,\ 10,\ 11,\ 12$$

The seven simultaneous linear algebraic equations in H_k may then be expressed in the matrix form,

$$[C]\{H\} = \{D\} \qquad , \qquad (12)$$

the coefficients for which are computed from the following equations:

$$C_{1k} = (T_{z_{k2}} - T_{z_{k1}} + \frac{r_2}{2\ell}\ T_{r_{k2}})\ P_k;$$

$$D_1 = -\sum_{k=8}^{12} C_{1k} H_k$$

$$C_{2k} = (T_{z_{k3}} - T_{z_{k1}} + \frac{r_3}{2\ell} T_{r_{k3}}) P_k;$$

$$D_2 = -\sum_{k=8}^{12} C_{2k} H_k$$

$$C_{3k} = (T_{z_{k6}} - T_{z_{k1}} + \frac{r_6}{2\ell} T_{r_{k6}}) P_k;$$

$$D_3 = -\sum_{k=8}^{12} C_{3k} H_k$$

$$C_{4k} = (T_{z_{k3}} - T_{z_{k4}}) P_k \quad ,$$

$$D_4 = -\sum_{k=8}^{12} C_{4k} H_k$$

$$C_{5k} = (T_{r_{k3}} - T_{r_{k4}}) P_k \quad ,$$

$$D_5 = -\sum_{k=8}^{12} C_{5k} H_k$$

$$C_{6k} = (T_{z_{k5}} - T_{z_{k6}})\, P_k \qquad ; D_6 = -\sum_{k=8}^{12} C_{6k} H_k$$

$$C_{7k} = (T_{r_{k5}} - T_{r_{k6}})\, P_k \qquad ; D_7 = -\sum_{k=8}^{12} C_{7k} H_k$$

DISCUSSION OF RESULTS

The computer program employed the Gauss-Jordan Reduction procedure to solve equations (12). All the numerical results for the sample problem are shown in Table IV. Note that there is an infinite variety of solutions available so long as five bar types may be specified arbitrarily at the outset. The particular choice of $H_8 \ldots\ldots H_{12}$ used in the sample problem was obviously not practical because the resulting solution for H_6 and H_7 turned out to be negative (a physical impossibility). It is obvious, however, that further work with the sample problem, although not warranted at this time, will lead to a large number of practical soultions.

The "best" solution in any particular case depends upon conditions of the problem which are not properly dwelt upon here. It would be judicious to select five bar types whose cross-sectional areas may most profitably be influenced by other important considerations. One might then explore how the solution for the other seven bars depends upon each of the five independent bars taken one at a time. If, for example, bars 8 through 12 were to be sized primarily by other considerations, one could solve for:

$$\partial H_k / \partial H_8,\ \partial H_k / \partial H_9,\ \ldots\ldots \partial H_k / \partial H_{12} \quad (k = 1,\ 2,\ \ldots\ldots 7)$$

This would be accomplished by solving the equations (12) five times,

Table IV

Numerical Results in the Sample Antenna Problem
(computer data rounded off to two decimal places)

F_{z_1}	F_{z_2}	F_{z_3}	F_{z_4}	F_{z_5}	F_{z_6}	F_{z_7}
.38	2.74	2.74	2.74	1.26	1.26	0
F_{r_1}	F_{r_2}	F_{r_3}	F_{r_4}	F_{r_5}	F_{r_6}	F_{r_7}
.04	.56	1.31	1.31	.79	.79	0
P_1	P_2	P_3	P_4	P_5	P_6	
-.32	1.94	-1.22	2.44	-2.42	-2.73	
P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	
-3.83	5.11	-4.52	-6.40	9.65	12.53	
H_1	H_2	H_3	H_4	H_5	H_6	
236.78	9.31	13.21	5.45	14.13	-3.99	
H_7	H_8	H_9	H_{10}	H_{11}	H_{12}	
-1.92	1.00	1.67	1.30	.92	.25	
Z_1/H_8	Z_2/H_8	Z_3/H_8	Z_4/H_8	Z_5/H_8	Z_6/H_8	Z_7/H_8
33.68	33.06	28.48	28.48	24.59	24.59	0
R_1/H_8	R_2/H_8	R_3/H_8	R_4/H_8	R_5/H_8	R_6/H_8	R_7/H_8
0	3.05	10.86	10.86	14.55	14.55	-23.01

Note: The force at each joint is taken normal to the dish surface, and the total axial force is arbitrarily set at 100 pounds.

each time with only one member of the group, $H_8 \ldots H_{12}$ non-zero. Then $H_1 \ldots H_7$ could each be expressed as a linear combination of $H_8 \ldots H_{12}$, which could be selected to keep all sizes physically practical, to restrain total deflections within reasonable bounds, to minimize total weight and/or to account for a host of other pertinent considerations.

EXTENSION OF ANALYSES TO DIFFERENT APPLICATIONS

1. Unsymmetrical Loading of the Antenna Truss

In developing equations for the general unsymmetrical loading condition, it is clearly necessary to distinguish each bar and loading point individually. The sheer magnitude of the problem then demands solution by means of a digital computer having a large storage capacity. For example, if there are j loading points, i points whose deflections are required and b bars, one can expect to encounter the following:

a. Three load coefficient matrices (for axial, radial and tangential loads) having b rows and a column for each point which is either a loading point or a point whose deflection is required.

b. Eighty-seven final equations relating the axial and radial deflections of all points on the surface of the dish so that the paraboloidal shape and focal length are preserved. Note that tangential deflections are not important as long as they are small. Each coefficient in the equations would be computed as the sum of b terms.

2. Other Structural Forms

The analytical approach already described applies to all linear, elastic structural forms. The influence function, therefore, must be expressed in terms of the unknown structural characteristics. When, as in the usual case, the influence function is so complex that it cannot be extracted from the integral equation, Eq (2), two alternatives are available:

a. Adopt a trial-and-error approach until the influence function is found which results in the prescribed form of the deformation; or,

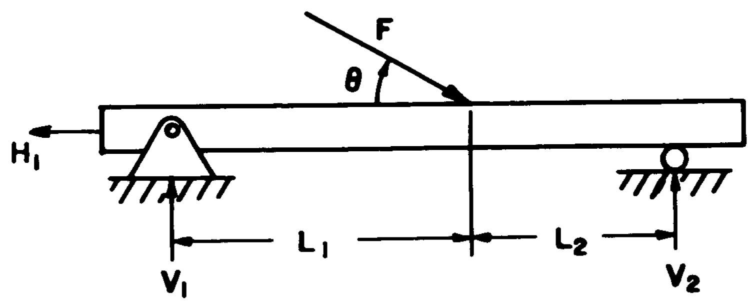
b. Adopt the matrix form of the integral equation and accept the limitation of prescribing only the deflection of several points. In principle this procedure leads to a direct solution for the influence coefficient matrices.

The relationships between the influence functions (or coefficients) and the strength characteristics of various structural forms such as beams, plates, etc. , are amply discussed in the literature. They are derived from the basic stress-strain differential equations by integrating the equations or by working with integrated forms of the equations (such as Clapeyron's Law).

3. Indeterminate Structures

Indeterminate structures are more complex to analyze because their deflections and internal loads must be obtained simultaneously. The following simple problem should illustrate this point. Fig. 2 illustrates a simple beam loaded by the external force, F, and supported in a determinate fashion by a pin at one end and a roller at the other. Static equilibrium clearly requires the satisfaction of three equations expressing the balance of forces and moments in the plane of the figure. The reactions H_1, V_1 and V_2 are then immediately obtained as are the internal loads (bending moment, shear force and axial force) at all stations along the beam.

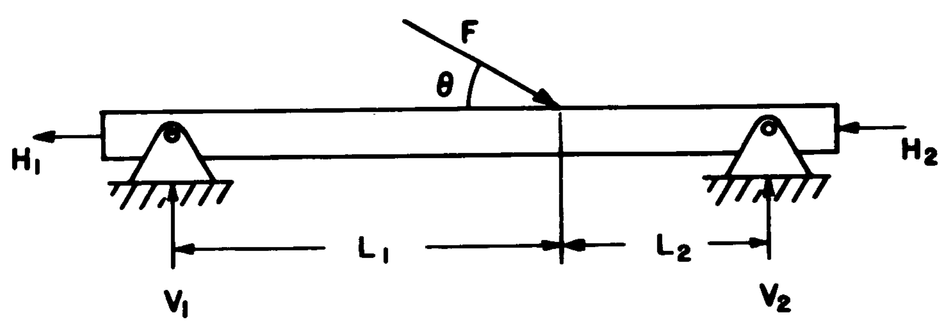
On the other hand, Fig. 3 illustrates an indeterminate beam supported by a pin at each end. In this case, static equilibrium considerations only give V_1, V_2 and the sum, $H_1 + H_2$. Even assuming the beam is pinned in place without pre-stressing it, one must consider the axial deflection of the beam before H_1, H_2 or the internal axial loads can be obtained.



STATIC EQUIL. REQUIRES

$$\begin{cases} H_1 = F\cos\theta \\ V_1 = \dfrac{L_2 F \sin\theta}{L_1 + L_2} \\ V_2 = \dfrac{L_1 F \sin\theta}{L_1 + L_2} \end{cases}$$

Fig. 2 A Determinate Beam

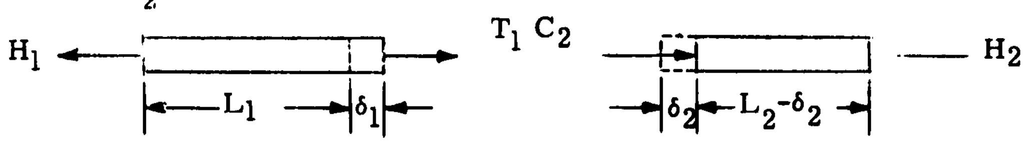


STATIC EQUIL. REQUIRES

$$\begin{cases} H_1 + H_2 = F\cos\theta \\ V_1 = \dfrac{L_2 F \sin\theta}{L_1 + L_2} \\ V_2 = \dfrac{L_1 F \sin\theta}{L_1 + L_2} \end{cases}$$

Fig. 3 An Indeterminate Beam

To do this, equate the stretching of the left end of the beam (of length L_1 and internal tensile force T_1) with the compression of the right end of the beam (of length L_2 and internal compressive force, C_2).



One then has the equations:

$$H_1 + H_2 = F \cos \theta$$

$$T_1 = H_1$$

$$C_2 = H_2$$

$$\delta_1 = T_1 L_1 \,/\, AE \qquad \Bigg\} \begin{array}{l} A = \text{beam cross-sectional area} \\ E = \text{Young's modulus} \end{array}$$

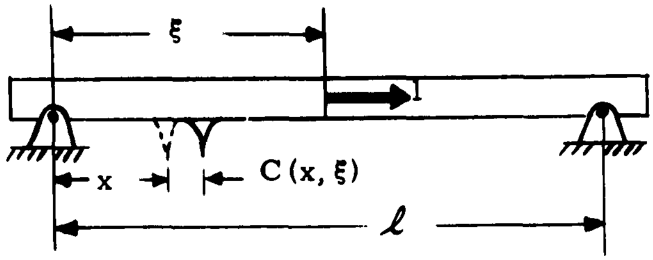
$$\delta_2 = C_2 L_2 \,/\, AE$$

$$\delta_1 = \delta_2$$

Solution of these equations yields (assuming no pre-loading):

$$H_1 = \frac{F \cos \theta}{1 + \dfrac{L_1}{L_2}} \qquad\qquad H_2 = \frac{F \cos \theta}{1 + \dfrac{L_2}{L_1}}$$

Note, however, that the computations necessary to determine the influence function relating axial deflection to external axial load have already been performed. If this influence coefficient is denoted by $C(x, \xi)$, and the following diagram is referred to,



one sees that:

$$C\,(x, \xi) = \frac{x\,(\ell - \xi)}{\ell AE} \qquad , \qquad 0 \leqq x \leqq \xi$$

$$C\,(x, \xi) = \frac{\xi\,(\ell - x)}{\ell AE} \qquad , \qquad \xi \leqq x \leqq \ell$$

When the right end is supported on a roller, as in the original determinate case:

$$C\,(x,\xi) = \frac{x}{AE} \qquad 0 \leqq x \leqq \xi$$

$$C\,(x, \xi) = \frac{\xi}{AE} \qquad \xi \leqq x \leqq \ell$$

This example bears out the previous contention that indeterminate structures are generally stiffer than determinate structures of the same size

In more complex examples one must employ more involved relationships among the deflections in order to reach a solution. For example, in the case of an indeterminate truss having welded joints, it is necessary to express the preservation of the angles between pairs of bars where they come together at the rigid joints.

The preceding discussion is intended to convey some appreciation of the manner in which one might go about analyzing an indeterminate structure for its influence functions. Expression of the physical restraints (or boundary conditions) on the deflections in one portion of the structure in terms of the deflections of the surrounding structure results in a set of equations, which, taken simultaneously with the equations of static equilibrium, define all the internal loads and structural deformations. From the standpoint of designing for prescribed deformations, the determinate structure has, in all but the simplest structural configurations, at least a distinct computational advantage.